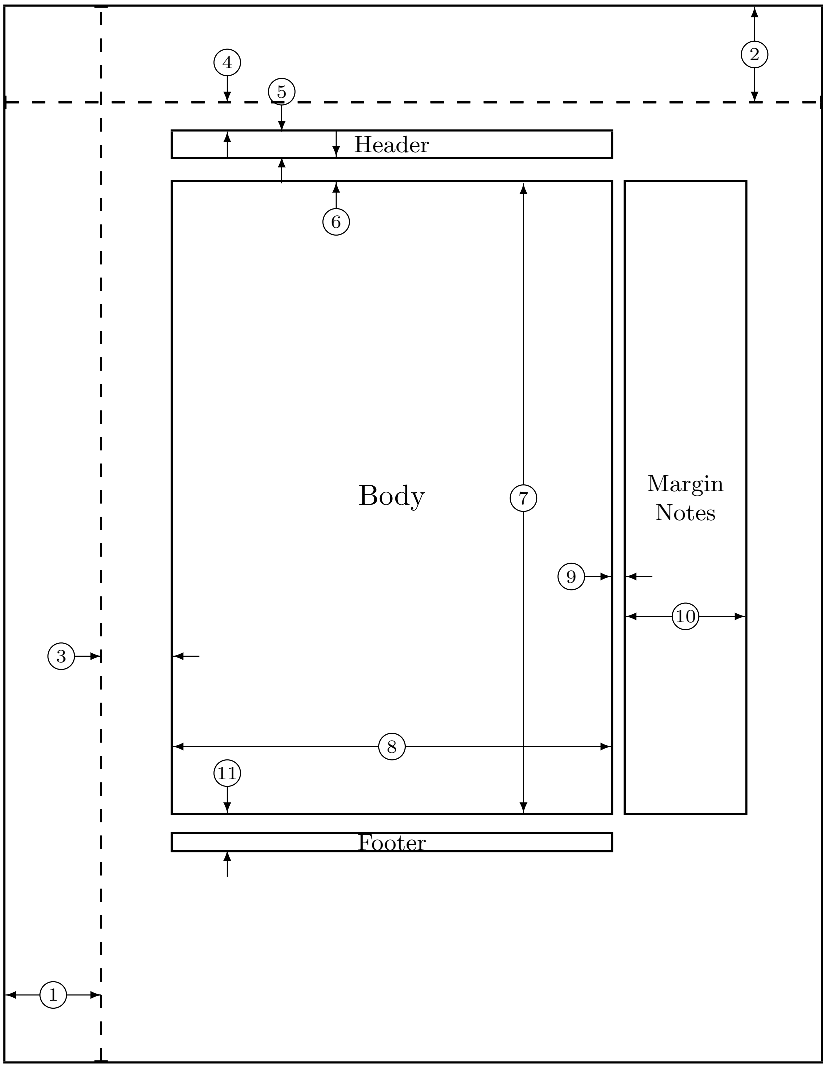

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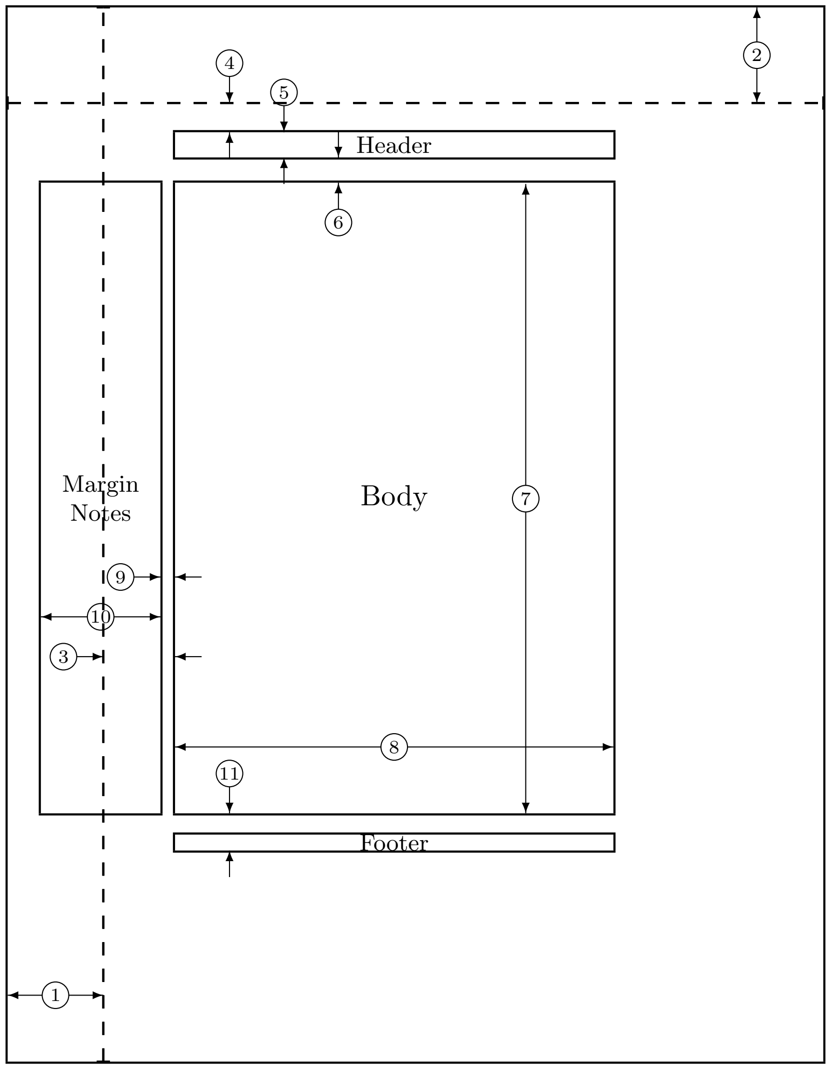

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RELATIONS BETWEEN TILTING AND STRATIFICATION.

Fidel Hernández Advíncula

Departament of Mathematic
Faculty of Mathemathic and Computer Sciences
Universidad de La Habana, Cuba

E-mail address: fidel@matcom.uh.cu

Abstract. In this work we study the relation between tilting and standard stratification. We recall that for each standardly stratified algebra corresponds a tilting module. We show that the poset given by the different stratifications of one algebra is a subposet of the poset formed by the tilting modules. Also, we show several examples, in particular we see that in the oriented A_n for $n = 2, 3, 4$ all tilting modules are given by stratifications.

Preliminars

In this work all algebras are finite dimensional K - algebras, basic and indecomposables, K is an algebraically closed field and it is known that an algebra Λ with these properties is of the form $\Lambda = \frac{KQ}{I}$ where Q is a finite quiver and I an admissible ideal.

Let $v_1, ...v_n$ be the vertices of Q in a fixed order and $S_1, ..., S_n$ the corresponding order of simple modules, P_i the projective cover of S_i and Q_i the injective envelope of S_i. The standard module Δ_i is defined as the maximal factor of P_i with composition factors $S_j, j \leq i$[R]. In dual way, it is defined the co-standard ∇_i as the maximal submodule of Q_i with composition factors $S_j, j \leq i$[R]

Let $\Delta = \{\Delta_1, ..., \Delta_n\}$, consider $F(\Delta)$, the full subcategory of mod Λ, consisting by $M \in$ mod Λ such that M has a filtration with factors in Δ, this is, $0 = M_0 \subset M_1 \subset ... \subset M_t = M$ con $\frac{M_i}{M_{i-1}} \simeq \Delta_k$. Dually, it is defined $F(\nabla)$.

There are the following subcategories of mod Λ:

- $Y(\Delta) = \{Y \in mod\Lambda / Ext^1(F(\Delta), Y) = 0\}$
- $F(\Delta) \cap Y(\Delta)$
- $W(\nabla) = \{W \in mod\Lambda / Ext^1(W, F(\nabla)) = 0\}$
- $W(\nabla) \cap F(\nabla)$

The algebra Λ is called standardly stratified if $\Lambda \in F(\Delta)$.

If also, the endomorphisms ring of each standard module is simple, Λ is called quasi - hereditary (see for instance [R] and [X]).

1. A tilting module associated to the standard stratification

An A - module T is called tilting (generalized) if:

(1) $pdT < \infty$.
(2) $Ext^i(T, T) = 0, \forall i > 0$
(3) There is an exact sequence $0 \to A \to T_0 \to T_1 \to .... \to T_s \to 0$, with $T_i \in addT, \forall i$.

If the algebra Λ is standardly stratified, we have that $F(\Delta)$ is a resolving category ([X]), i. e. is closed under extensions, kernel of surjections and contains the projectives.

Let $\varpi(\Delta)$ be the interseccion of the subcategories $F(\Delta)$ and $Y(\Delta)$

There is the following fact, proved in [X], Theor. 4.3:

Proposition 1. *If Λ is standardly stratified. Then there is a tilting module T, unique except for the multiplicity of the indecomposable direct summands such that $add(T) = \varpi(\Delta)$.*

2. A Poset given by the standard stratifications

For an Artin algebra Λ, consider the set $\mathcal{T}_\Lambda$ of all tilting modules with direct summands of multiplicity one.

For each tilting module $T \in \mathcal{T}_\Lambda$ consider the right perpendicular category $T^{\perp} = \{X \in mod\Lambda / Ext^i(T, X) = 0, \forall i\}$

In [HU], it is defined a partial order in the class of all tilting modules for an Artin algebra by the following relation $T_1 \leqslant T_2 \Leftrightarrow T_1^{\perp} \subseteq T_2^{\perp}$.

For this relation T is minimal if and only if $P^{<\infty}$ is contravariantly finite ([HU]).

Using the results of [AR], we see se that $Y(\Delta) = T^{\perp}$.

Theorem 2. *The order among the different forms in that an algebra can be standardly stratified , given by inclusion between the respective subcategories $F(\Delta)$, induces an inverse order between the tilting modules corresponding to these stratifications.*

Proof. If we have two orders of simple modules such that Λ is standardly stratified in these orders and $F_1(\Delta) \subset F_2(\Delta) \Rightarrow Y_2(\Delta) \subset Y_1(\Delta)$

(If $Y \in Y_2(\Delta) \Rightarrow Ext^1(X, Y) = 0, X \in F_2(\Delta)$, as $F_1(\Delta) \subset F_2(\Delta) \Rightarrow Ext^1(X, Y) = 0, X \in F_1(\Delta) \Rightarrow Y \in Y_1(\Delta)$).

Then we have $Y_2(\Delta) \subset Y_1(\Delta)$, and as $Y_i(\Delta) = T_i^{\perp}$ then $T_2^{\perp} \subseteq T_1^{\perp}$ $\square$

We know that $Proj \subset F(\Delta) \subset$ mod A, also $F(\Delta) \subset P^{<\infty}$.

If $F(\Delta) = P^{<\infty}$, that is to say $F(\Delta)$ is maximal then $P^{<\infty}$ is contravariantly finite, well $F(\Delta)$ it is, then T is minimal.

If $F(\Delta) = Proj$, that is to say $F(\Delta)$ is minimal then $Y(\Delta) = \{Y / Ext^1(X, Y) = 0, X \in F(\Delta)\} = modA$, then $F(\Delta) \cap Y(\Delta) = Proj$, therefore $T = P_1 \oplus ... \oplus P_n = A$, then $T^{\perp} = A^{\perp} = modA$ and we conclude that T is maximal.

If $F(\Delta)$ is maximal (minimal) not necessarily $F(\Delta) = P^{<\infty}(Proj)$

Example 3. *Let A_m be the algebra $\frac{KQ}{I}$ where Q is the quiver*

$$\begin{array}{ccccccccc} & \beta_1 & & \beta_2 & & & & \beta_{m-1} & \\ \overset{1}{\bullet} & \rightleftarrows & \overset{2}{\bullet} & \rightleftarrows & \overset{3}{\bullet} & \dots & \overset{m-1}{\bullet} & \rightleftarrows & \overset{m}{\bullet} \\ & \alpha_1 & & \alpha_2 & & & & \alpha_{m-1} & \end{array}$$

and I the ideal generated by $\alpha_{i+1}\alpha_i, \beta_i\beta_{i+1}, \alpha_i\beta_i - \beta_{i+1}\alpha_{i+1}$, $1 \leq i \leq m-2, \alpha_{m-1}\beta_{m-1}$.

We can see that this algebra is quasi hereditary, only in this order of simple modules, then $F(\Delta)$ is maximal and minimal because the poset has only one element and $F(\Delta) \neq P^{<\infty}$ and $F(\Delta) \neq Proj$.

3. Remarks and Examples

Remark 4. *We have several cases in that the maximal and the minimal are reached for the Poset given by the standard stratifications*[HM1]

(1) *The Hereditary algebras*
(2) *The quasi hereditary algebras without oriented cycles, except loops*
(3) *The algebras which are standardly stratified in all orders*

For the algebras with radical square zero, if it quasi triangular it is reached the minimal and the maximal.[HM2]

Remark 5. *For the hereditary algebras given by the quiver A_n for $n = 2, 3, 4$, we can check that all tilting modules are given by stratifications, but for the hereditary algebra given by the quiver*

$$\begin{tikzcd} 1\bullet \arrow[dr] & \\ & \bullet 3 \\ 2\bullet \arrow[ur] & \end{tikzcd}$$

the tilting module $T = P_1 \oplus P_2 \oplus I_3$ is not associated to stratification.

In the Kronecker algebra, that is to say the hereditary algebra given by the quiver $1\bullet \rightrightarrows \bullet 2$ we only have two stratifications: the one given by the projectives and the other given by the injectives and we have infinite tilting modules.

The algebra given by the quiver $1\bullet \rightleftarrows \bullet 2$ with radical square zero is not standardly stratified in any orden and we have an unique tilting module which is the trivial given by the sum of the projectives.

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